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Symmetry-protected topological phases of alkaline-earth cold fermionic atoms in one dimension

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Abstract – We investigate the existence of symmetry-protected topological phases in one-dimensional alkaline-earth cold fermionic atoms with general half-integer nuclear spin I at half-filling. In this respect, some orbital degrees of freedom are required. They can be introduced by considering either the metastable excited state of alkaline-earth atoms or the p-band of the optical lattice. Using complementary techniques, we show that SU(2) Haldane topological phases are stabilised from these orbital degrees of freedom. On top of these phases, we find the emergence of topological phases with enlarged SU$(2I+1)$ symmetry which depend only on the nuclear-spin degrees of freedom. The main physical properties of the latter phases are further studied using a matrix-product state approach. On the one hand, we find that these phases are symmetry-protected topological phases, with respect to inversion symmetry, when $I = 1/2, 5/2, 9/2, \ldots$, which is directly relevant to ytterbium and strontium cold fermions. On the other hand, for the other values of I(=half-odd integer), these topological phases are stabilised only in the presence of exact SU$(2I+1)$ symmetry.

Quantum phases of matter with exotic orderings have attracted much interest over the years. Prominent examples are topological phases which do not break any symmetry and cannot be characterised by local order parameters. Though all one-dimensional (1D) gapful phases have short-range entanglement, a stable topological phase can still be defined in 1D by the presence of certain symmetries which protect the phase [1]. A topological phase of this kind is called a symmetry-protected topological (SPT) phase [1,2]. Otherwise, it can be smoothly connected to a trivial gapful phase without any phase transition [3]. The Haldane phase [4] of the spin-1 Heisenberg chain displays striking properties which makes it a paradigmatic example of featureless gapped topological phases in 1D. For a long time, two features have been considered as the defining properties of the "topological" Haldane phase: the existence of hidden non-local string ordering [5] and the presence of a finite gap above a unique ground state (GS) with periodic boundary conditions while emergent spin-1/2 edge states are liberated at the boundaries of open chains [6]. However, the precise meaning as a SPT phase has been recognised quite recently [1] and it is now known that the Haldane phase is a relatively robust topological phase protected by the presence of at least one of the three discrete symmetries: the dihedral group of π rotations along the x, y, z axes, time-reversal and inversion symmetries [3,7]. Also, there are several proposals to realise the Haldane phase in cold-atom systems [8–11].

A recent breakthrough has led to the complete classification of SPT of 1D spin systems [12–15]. It relies on the determination of the projective representations of the underlying symmetry group which can be obtained using group cohomology. In particular, for spin systems with a high SU(N) symmetry, N distinct topological phases are

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expected from this classification [16]. A natural question is the physical realisation of these SU(N) topological phases starting from realistic fermionic systems.

In this respect, alkaline-earth like fermionic ultracold atoms seem to be very promising since they are the best candidates for experimental realisations of exotic high-symmetry many-body physics [17,18]. In this context, the main interest in those atoms stems from the presence of a GS $^1S_0$ ("g") and a metastable excited state $^3P_0$ ("e") between which transitions are forbidden. Moreover, both states have zero electronic angular momentum, so that the nuclear spin I is decoupled from the electronic spin. The nuclear-spin–dependent variation of the scattering lengths is expected to be smaller than $\sim 10^{-9}$ for the g state and $\sim 10^{-3}$ for the e state. This results in fermionic systems with an extended SU($N = 2I + 1$) symmetry [17]. Such gases of strontium (^{87}Sr) atoms ($I = 9/2$) or ytterbium (^{171}Yb, ^{173}Yb) atoms ($I = 1/2, 5/2$, respectively) have been cooled down to reach the quantum degeneracy [19–22].

If one only keeps the g state of the atoms, the effective low-energy Hamiltonian of this N-component Fermi gas boils down to the SU(N) Hubbard model [17]. However, the gapped Mott-insulating phases of the latter model loaded into a 1D optical lattice are known to be spatially non-uniform for all commensurate fillings [23]. In this respect, no 1D topological phases can be formed.

In this letter, we show, by means of complementary methods, that the situation is radically different if one includes orbital degrees of freedom. These degrees of freedom are naturally introduced here by either considering the metastable excited e state of alkaline-earth atoms [17] or the two-fold degenerate p-band of the 1D optical lattice [11]. The interplay between orbital and nuclear-spin degrees of freedom gives rise to different 1D insulating topological phases at half-filling (N atoms per site). Two different classes of topological phases are stabilised when N is even, *i.e.* I is a half-odd integer. We find Haldane phases with integer orbital pseudospin-$N/2$ and topological phases with enlarged SU(N) symmetry that we fully characterise. In particular, we investigate the topological protection of these phases with respect to the inversion symmetry by means of a matrix-product state (MPS) approach. It is shown that the topological phases of alkaline-earth cold fermions are SPT phases when $N/2$ is odd, *i.e.*, $I = 1/2, 5/2, 9/2, \ldots$, which is directly relevant to ytterbium and strontium atoms.

Strong-coupling approach. – The effective Hamiltonian which governs the low-energy properties of the problem is the two-orbital SU(N) Hubbard model: [11,17,24]

$$\mathcal{H} = -t\sum_{i,l\alpha}\left(c^\dagger_{l\alpha,\,i}c_{l\alpha,\,i+1} + \text{H.c.}\right) - \mu\sum_i n_i + \frac{U}{2}\sum_i n_i^2 + J\sum_i\left[(T_i^x)^2 + (T_i^y)^2\right] + J_z\sum_i (T_i^z)^2, \quad (1)$$

where $c^\dagger_{l\alpha,\,i}$ denotes the fermion creation operator at the i-th site with nuclear-spin states $\alpha = 1, \ldots, N$. The general orbital index $l = 1, 2$ stands either for the g and e states of alkaline-earth like fermionic atoms [17] or for the second double degenerate p orbitals, p_x and p_y, of the optical lattice [11]. In eq. (1), $n_i = \sum_{l\alpha} c^\dagger_{l\alpha,\,i}c_{l\alpha,\,i}$ denotes the occupation number on the site i, and $2\vec{T}_i = \sum_{lm\alpha} c^\dagger_{l\alpha,\,i}\vec{\sigma}_{lm}c_{m\alpha,\,i}$ is the orbital pseudospin operator ($\vec{\sigma}$ being the Pauli matrices). For the sake of simplicity, we restrict ourself here to the case where the two orbitals play a similar role. As it will be discussed below, all topological phases of the problem are robust against small terms which breaks the Z_2 orbital symmetry $1 \leftrightarrow 2$. On top of the U(1)$_c$ charge symmetry ($c_{l\alpha,\,i} \to e^{i\theta}c_{l\alpha,\,i}$), model (1) features an SU(N) symmetry ($c_{l\alpha,\,i} \to \sum_\beta U_{\alpha\beta}c_{l\beta,\,i}$, U being an SU(N) matrix), and a U(1)$_o$ symmetry in the orbital space spanned by the orbital states ($c_{1,2\,\alpha,\,i} \to e^{\pm i\theta}c_{1,2\,\alpha,\,i}$). Though model (1) enjoys an extended U(1)$_c$ × U(1)$_o$ × SU(N), the determination of its phase diagram is a highly non-trivial problem in the general case. However, one can already anticipate the existence of different topological phases by means of a strong-coupling analysis along special lines of model (1). These phases are protected by their spectral gaps and are stable against small symmetry-breaking perturbations. In this respect, let us consider the special case $J = J_z$ where the U(1)$_o$ orbital symmetry is enlarged to SU(2)$_o$. The single-site energy-spectrum at half-filling ($\mu = UN$) is labelled by two integers p, q:

$$E(p,q) = \frac{U}{2}(p+q)(p+q-2N) + \frac{J}{4}(p-q)(p-q+2),$$
$$D(p,q) = \frac{N!(N+1)!(p-q+1)^2}{(N-p)!(N+1-q)!(p+1)!q!}; \quad (2)$$

D is the degeneracy, $n = p + q$ is the number of fermions on one site, and $T = (p - q)/2$ is the spin of the orbital pseudospin operator $\vec{T}_i$. A first interesting line is $U = 0$ and $J < 0$ where the lowest-energy states (2) are $N + 1$ degenerate ($p = N$ and $q = 0$) and $\vec{T}_i$ is a $N/2$ pseudospin operator. At second order of perturbation theory in $|J| \gg t$, we find a pseudospin-$N/2$ antiferromagnetic SU(2) Heisenberg chain: $\mathcal{H}_{\text{eff}} = J_o\sum_i \vec{T}_i \cdot \vec{T}_{i+1}$ with $J_o = 8t^2/N(2N+1)|J|$. As is well known, the physics of the latter model strongly depends on the parity of N [4]. When N is odd, the phase is gapless with one gapless bosonic mode while, in the even N case, it is fully gapped (Haldane gap). We thus find the emergence of an SU(2) topological (Haldane) phase when N is even, where the SU(2) symmetry stems from the orbital degrees of freedom. We dub this phase *Haldane-orbital phase* (HO), since this phase is different from the Haldane phase for (nuclear) spin degrees of freedom (see fig. 1, where a cartoon represents this phase for $N = 2$). As is well known, these Haldane phases are robust against SU(2) symmetry-breaking perturbations. In addition, they are SPT phases only when $N/2$ is odd [7].

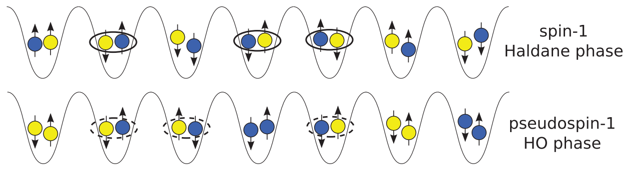

Fig. 1: (Colour on-line) For $N = 2$: the blue and yellow atoms describe the orbital states $l = 1, 2$. The Haldane phases for nuclear-spin and orbital (HO) degrees of freedom present a dilute order [5] that can be seen here as an alternation of sites with total (pseudo)spin component ± 1, diluted with an arbitrary number of sites with (pseudo)spin component 0.

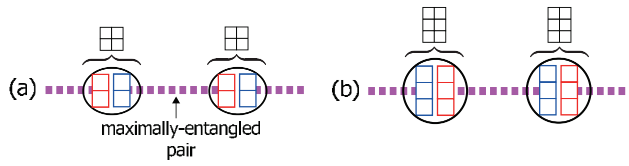

Fig. 2: (Colour on-line) SU(N) VBS states are constructed out of a pair of self-conjugate representations at each site. Dashed lines denote maximally entangled pairs. (a) SU(4) with 20-dimensional representation and (b) SU(6) with 175-dimensional representation.

A second interesting line is $J = 2NU/(N+2) > 0$ with even N. The lowest-energy states correspond to $p = q = N/2$ (N fermions which are orbital singlets) and transform into the SU(N) self-conjugate representations described by a Young tableau with two columns and $N/2$ rows. At second order of perturbation theory in $|U| \gg t$, we find an SU(N) Heisenberg spin chain in this representation. For $N = 2$, the spin-1 Haldane phase is formed among the nuclear spins and, in this respect, is intrinsically different from the HO phase with $N = 2$ (see fig. 1). When $N > 2$, the physical properties of the GS of the SU(N) magnet are not known. A confinement of spinons is expected from the general classification of ref. [25], and a non-degenerate gapful phase was predicted in the large-N limit [26].

Valence-bond-solid (VBS) approach. – To get a good insight into the properties of the GS of this SU(N) Heisenberg chain, we construct a series of model GS, the VBS states [27], whose parent Hamiltonian is close to the original one. We start from a pair of the self-conjugate representations (characterised by a Young tableau with *one* column and $N/2$ rows) on each site and create maximally entangled pairs between adjacent sites (see fig. 2). Last, we obtain the model VBS state by projecting the tensor-product states on each site onto the desired self-conjugate representation [28]. We explicitly constructed the matrix-product representation

$$\sum_{\{m_i\}} A_1(m_1)A_2(m_2)\cdots A_i(m_i)\cdots|m_1, m_2, \ldots, m_i, \ldots\rangle \tag{3}$$

of such a state for $N = 4$ (with the dimension of the A-matrices being 6 and $\{m_i\}$ labelling the 20 physical states at each site) to obtain "spin-spin" correlations exponentially decaying with correlation length $1/\ln 5 \approx 0.6213$. The parent Hamiltonian which supports the above VBS state as the exact GS is given by

$$\mathcal{H}_{\text{VBS}} = J_s \sum_i \Big\{ S_i^A S_{i+1}^A + \frac{13}{108}(S_i^A S_{i+1}^A)^2 + \frac{1}{216}(S_i^A S_{i+1}^A)^3 \Big\}, \tag{4}$$

where S_i^A denote the SU(4) spin operators in the 20-dimensional representation. We observe that model (4) is not very far from the original pure Heisenberg Hamiltonian (with spin exchange J_s) obtained by the t/U-expansion. This strongly suggests that an SU(4) topological phase is stabilised in the strong-coupling regime with the emergent edge states belonging to the 6-dimensional representation of SU(4). From the structure of the edge states, one can easily see that the VBS state found above belongs to one of the N topological classes protected by SU(N) symmetry [16].

It is interesting to check its robustness in the absence of SU(N) symmetry. Recently, it has been demonstrated that the even-fold degenerate structure in the entanglement spectrum is a fingerprint of the topological Haldane phase protected by inversion symmetry [3]. One may think that the $N = 4$ VBS state above represents a topologically robust Haldane state since its six finite entanglement eigenvalues are all degenerate. However, this even-fold degeneracy is accidental and the corresponding topological phase is protected only in the presence of high symmetries (*e.g.* SU(N)). In fact, if the phase is protected by the link-inversion symmetry, the unitary matrix U_I satisfying [29]

$$A^T(m) = e^{i\theta_I} {U_I}^{\dagger} A(m) U_I \tag{5}$$

should be antisymmetric [3]. For the $N = 4$ VBS state, $U_I^T = U_I$ and the state becomes trivial only in the presence of such an elementary symmetry as the link-inversion. Since the dimension of the self-conjugate representation is rapidly growing (1764 for SU(8)), it is not practical to explicitly construct the MPS for $N \geq 6$. However, it can be shown that the symmetry of U_I is determined solely by that of the maximally entangled (singlet) pair; if the latter pair is antisymmetric with respect to the interchange of the two constituent states, U_I is antisymmetric and the corresponding MPS remains topological even without high symmetries.

By investigating the form of the maximally entangled pair, we conclude that for $N/2$ odd the VBS state remains to be the stable topological Haldane phase protected by inversion symmetry while it is not the case for $N/2$ even. In particular, as in the HO case, we expect that the gapped SU(N) phase realised along the line $J = 2NU/(N+2)$ is SPT when $I = 1/2, 5/2, 9/2, \ldots$.

Low-energy approach. – The low-energy effective field theory of the lattice model (1) is derived by expressing the standard continuum limit of the lattice fermionic operators $c_{l\alpha\,i}$ in terms of $2N$ left- and right-moving $L_{l\alpha}$, $R_{l\alpha}$ Dirac fermions [30]: $c_{l\alpha\,i} \sim R_{l\alpha} e^{ik_F x} + L_{l\alpha} e^{-ik_F x}$, where $x = ia_0$, a_0 being the lattice spacing and $k_F = \pi/(2a_0)$ the Fermi momentum. The continuum Hamiltonian then takes the form of $2N$ Dirac fermions coupled with marginal four-fermions interactions. The low-energy properties of the resulting field theory are determined by means of a one-loop renormalization group (RG) analysis. This analysis has been done in the $N = 2$ case [31,32]. For instance, four different fully gapped Mott-insulating phases have been found when $J = J_z$. On top of conventional twofold degenerate phases with spin-Peierls (SP) and charge-density wave (CDW) orderings, the pseudospin-1 HO and the spin-1 Haldane phases, which have been identified in the strong-coupling analysis, persist in the weak-coupling regime as well.

The RG analysis in the general $N > 2$ case is much more involved. In some regions of the phase diagram, we find that the one-loop RG flow is attracted along two isotropic rays with SO($4N$) symmetry which is the maximal continuous symmetry achievable for $2N$ Dirac fermions. These highly symmetric rays signal the emergence of the CDW and SP phases in the general N case [33]. In sharp contrast to $N = 2$, the one-loop RG flow for $N > 2$ features a region with no symmetry restoration in the infrared (IR) limit. There is a separation of the energy scales: one of the perturbations, which depends only on the SU(N) spin degrees of freedom, reaches the strong-coupling regime faster than the others. A spin gap Δ_s is thus formed among the nuclear-spin degrees of freedom. Below the energy scale of the spin gap $E \ll \Delta_s$, the dominant part of the effective interacting Hamiltonian only involves the remaining charge and orbital degrees of freedom and, when $J = J_z$, it reads as follows:

$$\mathcal{H}_{\rm int}^{\rm eff} \simeq \lambda\,(\mathrm{Tr} g)^2 + \mu \cos\left(\sqrt{8\pi K_c/N}\,\Phi_c\right), \qquad (6)$$

where Φ_c is a Bose field which accounts for the U$(1)_c$ charge degrees of freedom, and K_c is its Luttinger parameter [30]. The low-energy properties of the charge degrees of freedom are captured by the sine-Gordon model at $\beta^2 = 8\pi K_c/N$ and we expect that the charge sector is gapped away since $K_c < N$. The IR properties of model (6) thus depend only on the orbital degrees of freedom and are described by the SU$(2)_N$ conformal field theory (CFT) perturbed by its spin-1 operator $(\mathrm{Tr} g)^2$ (g being the SU(2) matrix) with scaling dimension $4/(N+2)$. In this respect, the resulting low-energy field theory is exactly that of the spin-$N/2$ SU(2) Heisenberg chain derived by Affleck and Haldane using the non-Abelian bosonization approach [34]. We thus deduce the emergence of a Haldane-gap phase when N is even, *i.e.* for general half-integer nuclear spins. The resulting Haldane phase is identified with the HO phase which have been found already by the strong-coupling approach and is a collective singlet state formed among the orbital degrees of freedom.

In stark contrast, within the weak-coupling approach, we could see no evidence for a similar SU(N) topological phase of the nuclear spins for $N > 2$. In particular, along the $J = 2NU/(N+2) > 0$ line (N being even), a SP phase is found in the RG approach instead of an SU(N) topological phase expected from the strong-coupling argument. Therefore a quantum phase transition necessarily occurs at intermediate couplings and it is tempting to conjecture that the quantum critical point is described by an SU$(N)_2$ CFT. The situation may be understood as follows. First, one can determine by symmetry the low-energy field theory which is valid in the vicinity of this SU$(N)_2$ quantum critical point: $\mathcal{H}_{\rm eff} \simeq \mathcal{H}_{\mathrm{SU}(N)_2} + (g - g_c)|\mathrm{Tr}\ G|^2$, G being the SU$(N)_2$ primary field which belongs to the fundamental representation of SU(N). Following ref. [35], the nature of the phases of the latter model can be inferred from a semiclassical approach; when $g < g_c$, one has $\langle \mathrm{Tr} G\rangle \neq 0$, and the GS is twofold degenerate as a consequence of broken translation symmetry ($G \to -G$). This may be identified with the SP phase found in the weak-coupling limit. On the strong coupling side, when $g > g_c$, the semiclassical analysis now gives an SU(N) matrix with the constraint $\mathrm{Tr} G = 0$, and the phase is translationally invariant. The resulting effective field theory is known to be the Grassmannian sigma model on $U(N)/[U(N/2) \times U(N/2)]$ manifold with a $\theta = 2\pi$, topological theta term, which is massive [35]. The latter is known to be the semiclassical field theory of the SU(N) Heisenberg spin chain in self-conjugate representations with two columns [26]. We thus conclude that the SU(N) topological phase, identified within the VBS approach based on the strong-coupling Hamiltonian, emerges for $g > g_c$.

DMRG calculations. – A density-matrix renormalization group [36] (DMRG) is clearly called for to shed light on the existence of this quantum phase transition for moderate couplings. Model (1) was rewritten as an N-leg Hubbard ladder with additional rung interactions to get more efficient simulations. Typically, we used open boundary conditions and kept up to 1600 and 3200 states for convergence with $N = 2$ and $N = 4$, respectively, to get a discarded weight below 10^{-5}.

Figure 3(a) shows the bond kinetic energy for various $U > 0$ using $N = 4$ along the special line $J = 2NU/(N+2) = 4U/3$. As U increases, we clearly see a quantum phase transition from a SP phase to a uniform non-degenerate one. This is in agreement with our conjecture based on the low-energy and strong-coupling analysis. In order to confirm that the non-degenerate phase corresponds to the SU(4) topological phase discussed above, we plot in fig. 3(b) the local densities $n_{i,\alpha} = \sum_l c^\dagger_{l\alpha,\,i} c_{l\alpha,\,i}$ for each flavour $\alpha = 1,\ldots,4$ in this phase to investigate its edge states. Clearly, edge states are present with an exponential profile and a correlation length of the order of 3 lattice spacings for $U = 6$ and $J = 8$ (see fit). Moreover, DMRG data randomly show different kinds

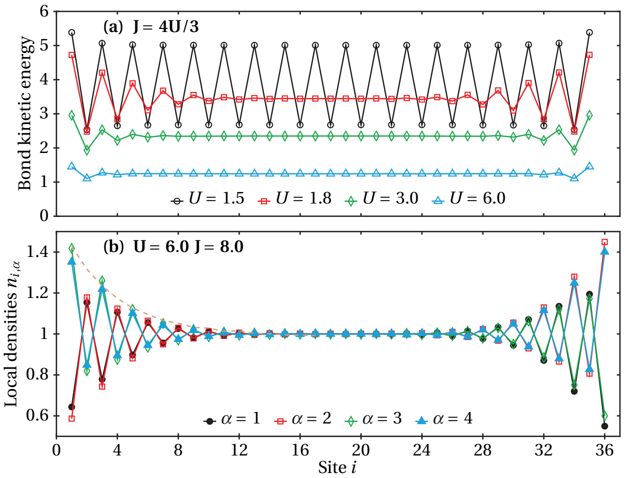

Fig. 3: (Colour on-line) (a) Bond kinetic energy for various $U > 0$ along the line $J = 4U/3$ ($N = 4$) in a $L = 36$ chain ($t = 1$). (b) Local densities $n_{i,\alpha}$ in the topological phase showing the existence of edge states. The dashed line is an exponential fit with a correlation length of around 3 lattice spacings.

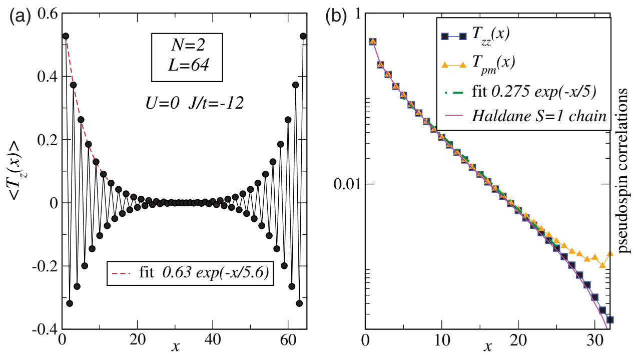

Fig. 4: (Colour on-line) $N = 2$ data for a chain of length $L = 64$ with $U = 0$ and $J/t = -12$: (a) local average $\langle T_z(x)\rangle$ *vs.* position x in the GS with total $T_z = 1$ showing evidence of localised edge states; (b) pseudospin correlations $T_{zz}(x) = \langle T_z(L/2)T_z(L/2+x)\rangle$ and $T_{pm}(x) = \langle T_+(L/2)T_-(L/2+x)\rangle/2$ in the GS with $T_z = 0$ exhibit an $SU(2)_o$ symmetry, are short-ranged and almost identical to spin correlations measured in spin-1 Heisenberg chain. Bulk and edge correlation lengths are close to 5 lattice spacings.

of edge states: in the sense that for a given edge, two flavours out of four give the same local densities [37]. Since there are six ways to choose two flavours out of four, our numerical data confirm that edge states are 6-fold degenerate in agreement with the VBS prediction that they belong to 6-dimensional SU(4) representation.

Another related evidence for the SU(N) topological phase comes from the degeneracy of the entanglement spectrum. According to the classification of the SU(N) SPT phases in [16], the SU(4) VBS state discussed above corresponds to one of the stable topological phases predicted there. Therefore, one may expect, on physical grounds, that the six-fold degeneracy of the lowest entanglement eigenvalue can be used as the fingerprint. To check this, we considered the ground-state corresponding to the parameters used in fig. 3(b) and measured the entanglement spectrum in the SU(4) topological phase. In fact, we observed that the lowest eigenvalue (*i.e.* the largest weight) is 6-fold degenerate as expected. The full characterisation of our SU(N) topological phase in terms of entanglement spectrum and the (non-local) order parameters [38–40] will be discussed in future work.

We also observed that these edge states are absent in the SP phase, and correspondingly the entanglement spectrum is non-degenerate. Therefore, this quantum phase transition is an example of topological one since the entanglement spectrum is totally different in each phase. However, the precise location of the phase transition and its universality class, as well as the full phase diagram, require extensive large-scale simulations and are thus beyond the scope of the letter. Finally, let us mention that our preliminary data indicate that this topological phase has a rather large extension in the phase diagram (a large part of the quadrant $U, J > 0$), which confirms its robustness and relevance for such parameters.

Now, we turn to the illustration of the HO phase for $N = 2$ since the case $N = 4$ would be quite demanding numerically [41]. Figure 4(a) shows the presence of edge states in a finite chain of length $L = 64$, similarly to what is found in the Haldane phase of the spin-1 chain. In fig. 4(b), we also plot pseudospin correlations (taken from the middle of the chain) that are short-ranged and almost identical to spin correlations measured in a spin-1 Heisenberg chain. Overall, we confirm the existence of the HO phase in this region.

Conclusion. – We have found that non-trivial topological phases can exist in 1D alkaline-earth cold atomic systems, thanks to the interplay between orbital and nuclear-spin degrees of freedom. More explicitly, we have shown the existence of i) the *Haldane-orbital phase*, which is the analogue of the usual Haldane phase but with orbital degrees of freedom, and also ii) an SU(N) topological phase for which we give an explicit construction using MPS approach and that possesses non-trivial edge states. Moreover, both topological phases correspond to the symmetry-protected Haldane phase provided $N/2$ is odd, which is directly relevant to strontium and ytterbium cold atoms, while the new SU(N) phase for the other even-N's is stable only in the presence of exact SU(N) symmetry.

In principle, experimental realisations of such phases could be achieved using alkaline-earth cold atoms such as ytterbium (^{171}Yb and ^{173}Yb) [21,22] and strontium (^{87}Sr) [19], that are known to realise SU(N) symmetry with great accuracy. In such a case, the orbital degree of freedom could be provided either by the metastable excited e state of the atoms [17], or by the p-band of the 1D optical lattice [11]. In a harmonic trap potential, the polarisation on the outer region of cold atomic systems can be measured [42,43] which may open perspectives for the observation of SPT in alkaline-earth cold atoms.

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REFERENCES

[1] Gu Z.-C. and Wen X.-G., *Phys. Rev. B*, **80** (2009) 155131.
[2] Chen X., Gu Z.-C. and Wen X.-G., *Phys. Rev. B*, **82** (2010) 155138.
[3] Pollmann F., Turner A. M., Berg E. and Oshikawa M., *Phys. Rev. B*, **81** (2010) 064439.
[4] Haldane F. D. M., *Phys. Lett. A*, **93** (1983) 464.
[5] den Nijs M. and Rommelse K., *Phys. Rev. B*, **40** (1989) 4709.
[6] Hagiwara M., Katsumata K., Affleck I., Halperin B. I. and Renard J. P., *Phys. Rev. Lett.*, **65** (1990) 3181.
[7] Pollmann F., Berg E., Turner A. M. and Oshikawa M., *Phys. Rev. B*, **85** (2012) 075125.
[8] García-Ripoll J. J., Martin-Delgado M. A. and Cirac J. I., *Phys. Rev. Lett.*, **93** (2004) 250405.
[9] Dalla Torre E. G., Berg E. and Altman E., *Phys. Rev. Lett.*, **97** (2006) 260401.
[10] Nonne H., Lecheminant P., Capponi S., Roux G. and Boulat E., *Phys. Rev. B*, **81** (2010) 020408(R).
[11] Kobayashi K., Okumura M., Ota Y., Yamada S. and Machida M., *Phys. Rev. Lett.*, **109** (2012) 235302.
[12] Chen X., Gu Z.-C. and Wen X.-G., *Phys. Rev. B*, **83** (2011) 035107.
[13] Chen X., Gu Z.-C. and Wen X.-G., *Phys. Rev. B*, **84** (2011) 235128.
[14] Schuch N., Pérez-García D. and Cirac I., *Phys. Rev. B*, **84** (2011) 165139.
[15] Fidkowski L. and Kitaev A., *Phys. Rev. B*, **83** (2011) 075103.
[16] Duivenvoorden K. and Quella T., *Phys. Rev. B*, **87** (2013) 125145.
[17] Gorshkov A. V., Hermele M., Gurarie V., Xu C., Julienne P. S., Ye J., Zoller P., Demler E., Lukin M. D. and Rey A. M., *Nat. Phys.*, **6** (2010) 289.
[18] Cazalilla M. A., Ho A. F. and Ueda M., *New J. Phys.*, **11** (2009) 103033.
[19] DeSalvo B. J., Yan M., Mickelson P. G., Martinez de Escobar Y. N. and Killian T. C., *Phys. Rev. Lett.*, **105** (2010) 030402.
[20] Fukuhara T., Takasu Y., Kumakura M. and Takahashi Y., *Phys. Rev. Lett.*, **98** (2007) 030401.
[21] Taie S., Takasu Y., Sugawa S., Yamazaki R., Tsujimoto T., Murakami R. and Takahashi Y., *Phys. Rev. Lett.*, **105** (2010) 190401.
[22] Taie S., Yamazaki R., Sugawa S. and Takahashi Y., *Nat. Phys.*, **8** (2012) 825.
[23] Szirmai E., Legeza O. and Sólyom J., *Phys. Rev. B*, **77** (2008) 045106.
[24] Xu C., *Phys. Rev. B*, **81** (2010) 144431.
[25] Rachel S., Thomale R., Führinger M., Schmitteckert P. and Greiter M., *Phys. Rev. B*, **80** (2009) 180420.
[26] Read N. and Sachdev S., *Nucl. Phys. B*, **316** (1989) 609.
[27] Affleck I., Kennedy T., Lieb E. H. and Tasaki H., *Phys. Rev. Lett.*, **59** (1987) 799.
[28] These SU(N) VBS are different from the ones considered in the following references: Greiter M. and Rachel S., *Phys. Rev. B*, **75** (2007) 184441; Korepin V. and Xu Y., *Int. J. Mod. Phys. B*, **24** (2010) 1361; Orús R. and Tu H.-H., *Phys. Rev. B*, **83** (2011) 201101(R).
[29] Pérez-García D., Wolf M. M., Sanz M., Verstraete F. and Cirac J. I., *Phys. Rev. Lett.*, **100** (2008) 167202.
[30] Gogolin A. O., Nersesyan A. A. and Tsvelik A. M., *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, UK) 1998.
[31] Nonne H., Boulat E., Capponi S. and Lecheminant P., *Mod. Phys. Lett. B*, **25** (2011) 955.
[32] Nonne H., Boulat E., Capponi S. and Lecheminant P., *Phys. Rev. B*, **82** (2010) 155134.
[33] Nonne H., Lecheminant P., Capponi S., Roux G. and Boulat E., *Phys. Rev. B*, **84** (2011) 125123.
[34] Affleck I. and Haldane F. D. M., *Phys. Rev. B*, **36** (1987) 5291.
[35] Affleck I., *Nucl. Phys. B*, **305** (1988) 582.
[36] White S. R., *Phys. Rev. Lett.*, **69** (1992) 2863.
[37] The same phenomenon occurs in the Haldane phase of spin-1 chain where a DMRG simulation done on a long enough chain in the $S_z^{tot} = 0$ sector will generate arbitrarily an edge state with $S_z = \pm 0.5$ at one boundary. Indeed, DMRG algorithm is known to select a minimally entangled state from the quasi-degenerate ground states in a topological phase. See Jiang H.-C., Wang Z. and Balents L., *Nat. Phys.*, **8** (2012) 902.
[38] Haegeman J., Pérez-García D., Cirac I. and Schuch N., *Phys. Rev. Lett.*, **109** (2012) 050402.
[39] Pollmann F. and Turner A. M., *Phys. Rev. B*, **86** (2012) 125441.
[40] Duivenvoorden K. and Quella T., *Phys. Rev. B*, **86** (2012) 235142.
[41] Our strong-coupling argument indicates that the HO phase of $N = 4$ is similar to a spin-2 Heisenberg chain, whose correlation length is very large ($\xi \sim 50$). See Schollwöck U., Golinelli O. and Jolicœur T., *Phys. Rev. B*, **54** (1996) 4038.
[42] Partridge G. B., Li W., Kamar R. I., Liao Y.-A. and Hulet R. G., *Science*, **311** (2006) 503.
[43] Liao Y.-A., Rittner A. S. C., Paprotta T., Li W., Partridge G. B., Hulet R. G., Baur S. K. and Mueller E. J., *Nature*, **467** (2010) 567.